AD-A042 707 WISCONSIN UNIV MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
DISCRETE EXPONENTIAL ABEL-EULER SPLINES.(U)
APR 77 R N MOHAPATRA, A SHARMA DAAG29-75-C-0024
UNCLASSIFIED MRC-TSR-1746 NL

1 OF 1
ADA042-707

END
DATE
FILMED
9-77
DDC

120

ADA042707

MRC Technical Summary Report #1746

DISCRETE EXPONENTIAL ABEL-EULER SPLINES

R. N. Mohapatra and A. Sharma

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

April 1977

Received April 20, 1977

Approved for public release
Distribution unlimited

See Form 1473

Sponsored by

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

DISCRETE EXPONENTIAL ABEL-EULER SPLINES

R. N. Mohapatra and A. Sharma

Technical Summary Report # 1746
April 1977

ABSTRACT

We study the class of functions $S_n(x)$ $(-\infty < x < \infty)$ having the following properties:

(i) The restriction of $S_n(x)$ to every unit interval $(\nu,\nu+1)(\nu\in\mathbb{Z})$ is a polynomial $S_{n,\nu}(x)$ of degree not exceeding n.

(ii) If $h > 0$ and $n\,h < 1$, and $S_{n,\nu}(x)$, $S_{n,\nu+1}(x)$ are two successive polynomials, then they satisfy the Abel interpolatory conditions

$$S_{n,\nu}^{(k)}(\nu + 1 + kh) = S_{n,\nu+1}^{(k)}(\nu + 1 + kh) \qquad (k = 0,1,\ldots,n-1).$$

These are a kind of discrete splines that may be called cardinal Abel splines. It is shown that the theory of the usual cardinal polynomial splines of degree n (See [6]) carries over to the new class of cardinal Abel Splines. The latter class reduces to the former if we let $h \to 0$. We obtain in this way generalizations of the exponential Euler polynomials of the exponential Euler splines, eigensplines a.s.f.

AMS(MOS) Subject Classification - 41A05, 41A15

Key Words: Cardinal splines, Abel interpolation

Work Unit #6 - Spline Functions and Approximation Theory

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

DISCRETE EXPONENTIAL ABEL-EULER SPLINES

R.N. Mohapatra and A. Sharma

1. Introduction. Discrete splines made their appearance first in the literature when Mangasarian and Schumaker [2] brought out their minimization properties and their relations with summation formulae. Later Tom Lyche [1] made a systematic study of the subject in his dissertation. More recently Tzimbalario [8] has studied discrete cardinal splines and has brought out an analogue of Schoenberg's exponential Euler splines. He has shown that the methods of Schoenberg can be adapted to solve the interpolation problem to data of power growth by discrete splines with the same power growth and having integer knots.

One of the significant features of the discrete splines of degree n so far available is that two polynomial components of a discrete spline of degree n on two adjacent intervals determined by three successive knots, say $x_{\nu-1}$, x_ν, $x_{\nu+1}$, agree in n equidistant points $x_\nu + kh$, $k = 0,1,\cdots,n-1$. This requirement is equivalent to requiring the first n differences of the polynomials on these n points to vanish. Thus the two polynomial components differ only by an expression $c \prod_{k=o}^{n-1} (x - x_\nu - kh)$ where c is a constant.

It appears that there is no special merit attached to this particular constraint. The object of this paper is to replace this requirement by another one related to the Abel interpolation problem. It turns out that the analogues of Schoenberg's result on Cardinal splines come out equally easily in this case as well.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

It also follows from this treatment that there can be many discrete splines depending upon the nature of interpolatory restraints which weld the two adjacent polynomial components. The discrete splines considered by Mangasarian and Schumaker [2] and Lyche [1] give one example of this observation.

In Section 2, we give the preliminaries and formulate two problems of interpolation. We also define the B-splines $Q^h_{n+1}(x)$ which form a basis for the discrete splines studied here. Section 3 deals with exponential Abel-Euler polynomials and their properties. We introduce the polynomials $\pi_{n,h/n}(x,\lambda)$ and show that for $h < x < 1$, these polynomials have real simple negative zeros. In Section 4 we use the results of Section 3 to define the exponential Abel-Euler spline and study its convergence as its degree tends to infinity. In Section 5 we sketch the outlines of a solution of Problem A of Section 2. We define t-perfect discrete Abel splines in Section 6 in analogy with the study of Sharma and Tzimbalario [7]. We propose the general problem and solve it for $r = n - 1$. For $r \leq n - 2$, we are not yet able to resolve the problem.

2. Preliminaries. Let π_n denote the set of polynomials of degree $\leq n$ and let P_n denote the class of functions $S_n(x)$ whose restriction $S_{n,\nu}(x)$ to $(\nu,\nu+1)$, for all integers ν, belongs to π_n. For a given real positive number h with $nh < 1$, we denote by S^h_n the class of functions $S_n(x)$ which satisfy the following conditions:

(i) $S_n(x) \in P_n$,

and for $\nu = 0, \pm 1, \pm 2, \cdots$,

(ii) $S^{(k)}_{n,\nu}(\nu+1+kh) = S^{(k)}_{n,\nu+1}(\nu+1+kh)$, $k = 0,1,\cdots,n-1$,

where $S_{n,\nu}(x)$ and $S_{n,\nu+1}(x)$ denote the restrictions of $S_n(x)$ to $(\nu,\nu+1)$ and $(\nu+1,\nu+2)$ respectively.

We shall call S^h_n the class of Cardinal Abel-discrete splines. As $h \to 0$, S^h_n reduces to the class of Cardinal splines in the sense of Schoenberg [6].

Following Schoenberg [6], we propose

Problem A. *Let* $\gamma \geq 0$, $0 \leq \alpha < 1$ *and let* $y = \{y_\nu\}$, $\nu = 0,\pm1,\cdots$, *be an arbitrary infinite sequence*

(2.1) $$|y_\nu| = O(|\nu|^\gamma).$$

Find a function $S_n(x) \in S^h_n$ *such that*

(2.2) $$S_n(\nu+\alpha) = y_\nu, \quad \nu = 0,\pm1,\cdots,$$

(2.3) $$|S_n(x)| = O(|x|^\gamma) \quad \textit{as} \quad x \to \pm\infty.$$

If $y_\nu = \lambda^\nu$, $\lambda \neq 0,1$, then the data is not of power growth and leads to Problem B.

Problem B. *For a given* α, $0 < \alpha < 1$, *find a function* $S_n(x) \in S^h_n$ *which satisfies the conditions:*

(2.4) $$S_n(\nu+\alpha) = \lambda^{\nu+\alpha}, \quad \nu = 0,\pm1,\cdots,$$

(2.5) $$S_n(x+1) = \lambda\, S_n(x).$$

We shall first turn to the solution of Problem B which in turn will lead to the solution of Problem A. We shall require the fundamental polynomials of

Abel-Gontcharoff interpolation:

(2.6) $$G_{0,h}(x) = 1,\quad G_{1,h}(x) = x,\quad G_{n,h}(x) = x(x-nh)^{n-1}\quad (n = 2,3,\cdots).$$

It is easy to verify that

(2.6a) $$G^{(k)}_{n,h}(x) = n(n-1)\cdots(n-k+1)G_{n-k,h}(x-kh),\quad k \leq n.$$

Set

(2.7) $$Q^h_{n+1}(x) = \frac{1}{n!}\sum_{j=o}^{n+o}(-1)^j\binom{n+1}{j}\left(G_{n,h}(x-j)\right)_+$$

where

$$\left(G_{n,h}(x)\right)_+ = \begin{cases} 0 & (x < 0), \\ G_{n,h}(x) & (x \geq 0). \end{cases}$$

It is easy to see that $Q^h_{n+1}(x) \in S^h_n$ and has the support $[0,n+1]$. Following Schoenberg [6] it can be shown that every $S(x) \in S^h_n$ has a unique representation in terms of $Q^h_{n+1}(x)$ and its translates, i.e.,

(2.8) $$S(x) = \sum_{j=-\infty}^{\infty} c_j\, Q^h_{n+1}(x-j).$$

In view of (2.8), the Problem B can be easily solved. Indeed

(2.9) $$S_n(x) = c_o \sum_{j=-\infty}^{\infty} \lambda^j\, Q^h_{n+1}(x-j)$$

satisfies the relation (2.5), and (2.4) will be satisfied if

(2.10) $$c_o \sum_{-\infty}^{\infty} \lambda^j\, Q^h_{n+1}(\alpha-j) = 1.$$

We shall show in Section 4 that for $nh < \alpha < 1$,

(2.11) $$\sum_{-\infty}^{\infty} \lambda^j\, Q^h_{n+1}(\alpha-j) \neq 0.$$

However the representation (2.9) is not satisfactory for the study of the convergence problem. We shall obtain another representation in the next section.

3. The Exponential Abel-Euler Polynomials. We shall consider the monic polynomials $A_{n,h}(x,\lambda)$ of degree n in x which are determined by the conditions:

$$(3.1)\qquad A_{n,h}^{(k)}(1+kh,\lambda) = \lambda\, A_{n,h}^{(k)}(kh,\lambda) \quad (k = 0,1,\cdots,n-1).$$

We shall call these polynomials exponential Abel-Euler polynomials. Since the polynomial $G_{n,h}(x)$ is of exact degree n, the polynomial $A_{n,h}(x,\lambda)$ is represented by

$$A_{n,h}(x,\lambda) = G_{n,h}(x) + \binom{n}{1}c_1\, G_{n-1,h}(x) + \cdots + c_{n-1}G_{1,h}(x) + c_n G_{o,h}(x).$$

The conditions (3.1) lead to the following system of equations in the c_ν's on using (2.6a):

$$(3.2)\qquad G_{\nu,h}(1) + \binom{\nu}{1}c_1 G_{\nu-1,h}(1) + \cdots + \binom{\nu}{\nu-1}c_{\nu-1}G_{1,h}(1) + c_\nu = \lambda c_\nu \quad (\nu = 1,2,\cdots,n).$$

It follows from (3.2) that if $\lambda \neq 1$, the c_ν's are uniquely determined.

In order to obtain some simple properties of the polynomials $A_{n,h}(x,\lambda)$ we observe that these polynomials are given by the generating function:

$$(3.3)\qquad \frac{\lambda - 1}{\lambda - e^z}\, e^{xz} = \sum_{n=o}^{\infty} \frac{A_{n,h}(x,\lambda)}{n!}\, z^n e^{nzh} .$$

This follows from (3.2) on observing that the sequence $\{c_\nu\}$ is independent of n and hence (3.2) gives

$$\lambda \sum_{\nu=1}^{\infty} c_\nu \frac{t^\nu}{\nu!} = \sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu!}\left\{G_{\nu,h}(1) + \sum_{k=1}^{\nu}\binom{\nu}{k}G_{\nu-k,h}(1)c_k\right\} .$$

Setting $c_o = 1$,

$$f(t) = \sum_{\nu=o}^{\infty} c_\nu \frac{t^\nu}{\nu!}, \quad g(t) = \sum_{n=o}^{\infty} G_{n,h}(1) \frac{t^n}{n!},$$

we get

$$f(t) = \frac{\lambda - 1}{\lambda - g(t)}. \tag{3.4}$$

Putting $t = ze^{zh}$ in (3.4) we easily get

$$\frac{\lambda-1}{\lambda - e^z} = \sum_{n=o}^{\infty} c_n \frac{z^n e^{nzh}}{n!}. \tag{3.4a}$$

It is known and is also easy to verify that

$$e^{xz} = 1 + xze^{zh} + \cdots + \frac{x(x-nh)^{n-1}}{n!} z^n e^{nzh} + \cdots \qquad (|zh| < 1). \tag{3.5}$$

Formula (3.3) now follows on multiplying (3.4a) and (3.5) and using Cauchy product on the right

We formulate some properties of $A_{n,h}(x,\lambda)$ in

THEOREM 1. *The polynomials* $A_{n,h}(x,\lambda)$ *satisfy the following relations:*

$$A_{n,h}(x,\lambda) = (1-\lambda) \sum_{\nu=o}^{\infty} \lambda^\nu G_{n,h}(x-\nu-1), \tag{3.6}$$

$$A_{n,h}\left(x, \frac{1}{\lambda}\right) = (-1)^n A_{n,-h}(1-x,\lambda), \tag{3.7}$$

$$A_{n,h}(1+x,\lambda) - \lambda A_{n,h}(x,\lambda) = (1-\lambda)G_{n,h}(x), \tag{3.8}$$

$$D_x A_{n,h}(x+h,\lambda) = n A_{n-1,h}(x,\lambda). \tag{3.9}$$

<u>Proof</u>: We observe that from (3.3) we have on using (3.5),

$$\sum_{n=0}^{\infty} A_{n,h}(x,\lambda)\,\frac{z^n e^{nzh}}{n!} = \frac{1-\lambda}{1-\lambda e^{-z}}\, e^{(x-1)z}$$

$$= (1-\lambda)\sum_{\nu=0}^{\infty} \lambda^{\nu}\, e^{(x-\nu-1)z}$$

$$= (1-\lambda)\sum_{\nu=0}^{\infty} \lambda^{\nu} \sum_{n=0}^{\infty} G_{n,h}(x-\nu-1)\,\frac{z^n e^{nzh}}{n!}$$

$$= (1-\lambda)\sum_{n=0}^{\infty} \frac{z^n e^{nzh}}{n!} \sum_{\nu=0} \lambda^{\nu}\, G_{n,h}(x-\nu-1).$$

This proves (3.6).

(3.7) is immediate from (3.3).

From the identity

$$\frac{\lambda-1}{\lambda-e^{z}} \cdot e^{(x+1)z} = -(\lambda-1)e^{xz} + \frac{\lambda(\lambda-1)}{\lambda-e^{z}}\, e^{xz}$$

we get (3.8) on using (3.3) and (3.5).

Lastly (3.9) also immediately follows from (3.3) by easy manipulation.

We now set

(3.10) $\quad \Pi_{n,h}(x,\lambda) = (\lambda-1)^n A_{n,h}(x,\lambda).$

It follows from (3.3) that $\Pi_{n,h}(x,\lambda)$ is a polynomial of degree n in λ and that

(3.11) $\quad \Pi_{n,h}(x,\lambda) = \lambda^n G_{n,h}(x) + \cdots + (-1)^n G_{n,h}(x-1).$

In particular we have

$$\Pi_{0,h}(x,\lambda) = 1 \qquad \Pi_{1,h}(x,\lambda) = \lambda x + (1-x)$$

$$\Pi_{2,h}(x,\lambda) = \lambda^2\, x(x-2h) - \lambda\{2x(x-2h) - 2x - 1 + 2h\}$$
$$+ (x-1)(x-1-2h).$$

If $x = 0$, $\Pi_{n,h}(0,\lambda)$ is a polynomial of degree $n-1$ in λ.

From Theorem 1 we can now get

COROLLARY 1. *The following representations for* $\Pi_{n,h}(x,\lambda)$ *are valid:*

(3.12) $$\Pi_{n,h}(x,\lambda) = (-1)^n(1-\lambda)^{n+1} \sum_{\nu=o}^{\infty} \lambda^\nu\, G_{n,h}(x-\nu-1),$$

(3.13) $$\Pi_{n,h}(x,\lambda) = n! \sum_{\mu=o}^{n} \lambda^\mu\, Q_{n+1}^{-h}(\mu+1-x).$$

Relations (3.12) follow immediately from (3.6). It follows from (2.7) that

$$Q_{n+1}^{h}(x) = Q_{n+1}^{-h}(n+1-x).$$

Hence from (3.12), we have

$$\Pi_{n,h}(x,\lambda) = (-1)^n \sum_{i=o}^{n+1} (-1)^i \binom{n+1}{i} \sum_{\nu=o}^{\infty} \lambda^{\nu+i}\, G_{n,h}(x-\nu-1)$$

$$= \sum_{i=o}^{n+1} (-1)^i \binom{n+1}{i} \sum_{-\infty}^{\infty} \lambda^{\nu+i}\, \left(G_{n,-h}(\nu+1-x)\right)_+$$

which easily reduces to (3.13).

We now formulate

THEOREM 2. *The polynomial* $\Pi_{n,\frac{h}{n}}(x,\lambda)$ *for a fixed* x, $h < x < 1$, *as polynomial in* λ *has* n *real negative simple zeros.*

Proof: The proof of this result depends upon the following relation:

$$(3.14)\quad \Pi_{n+1,\frac{nh}{n+1}}(x,\lambda) = \lambda(1-\lambda)D_\lambda \Pi_{n,h}(x,\lambda) + \{(n+1)\lambda - (1-\lambda)(x-1-nh)\}\Pi_{n,h}(x,\lambda)$$

To prove (3.14), we differentiate (3.12) with respect to λ so that

$$(3.14a)\quad D_\lambda \Pi_{n,h}(x,\lambda) = -\frac{n+1}{1-\lambda}\Pi_{n,h}(x,\lambda) + K_1 ,$$

where

$$K_1 = (-1)^n \frac{(1-\lambda)^{n+1}}{\lambda} \sum_{\nu=o}^{\infty} \nu\, \lambda^\nu\, G_{n,h}(x-\nu-1).$$

Writing $\nu\lambda^\nu = \{(x-1-nh) - (x-\nu-1-nh)\}\lambda^\nu$ in K_1 and observing that

$$(x-nh)G_{n,h}(x) = G_{n+1,\frac{nh}{n+1}}(x),$$

we have

$$K_1 = \frac{1}{\lambda}(x-1-nh)\Pi_{n,h}(x,\lambda) - \frac{1}{\lambda(1-\lambda)}\Pi_{n+1,\frac{nh}{n+1}}(x,\lambda).$$

From this (3.14a) we get (3.14).

Replacing h by $\frac{h}{n}$, we have from (3.14),

$$\lambda(1-\lambda)D_\lambda \Pi_{n,\frac{h}{n}}(x,\lambda) + \{(n+1)\lambda - (1-\lambda)(x-1-h)\}\Pi_{n,\frac{h}{n}}(x,\lambda) = \Pi_{n+1,\frac{h}{n+1}}(x,\lambda).$$

We observe that

$$(3.15)\quad \begin{cases} \operatorname{sgn} \Pi_{n+1,\frac{h}{n+1}}(x,0) = 1, \\[2ex] \operatorname{sgn} \Pi_{n+1,\frac{h}{n+1}}(x,-\Lambda) = (-1)^{n+1} \quad \text{when } x > h, \quad (\Lambda > 0 \text{ large}). \end{cases}$$

If $\lambda_n < \lambda_{n-1} < \cdots < \lambda_1 < 0$ denote the negative real simple zeros of $\Pi_{n,\frac{h}{n}}(x,\lambda)$, then

$$\lambda_\nu(1-\lambda_\nu)[D_\lambda \Pi_{n,\frac{h}{n}}(x,\lambda)]_{\lambda=\lambda_\nu} = \Pi_{n+1,\frac{h}{n+1}}(x,\lambda_\nu).$$

From this the result follows by induction on n on using (3.15).

<u>Remark</u>. Corresponding to each zero λ_ν of $\Pi_{n,\frac{h}{n}}(\alpha,\lambda)$ for $h \leq \alpha < 1$, we have a discrete Euler spline $S_\nu(x)$ which will satisfy the conditions:

$$S_\nu(\alpha+k) = 0, \quad k = 0,\pm1,\pm2,\cdots,$$

and

$$S_\nu(x+1) = \lambda_\nu \, S_\nu(x).$$

These discrete Euler splines have exponential growth at $+\infty$ or $-\infty$ which depends upon $|\lambda_\nu| > 1$ or < 1. These n discrete Euler splines form a basis of the class $\mathcal{S}^o_{n,h}$ of discrete splines where

$$\mathcal{S}^o_{n,h} = \{S(x)\,|\,S(x) \in \mathcal{S}^h_n, \quad S(\alpha+k) = 0, \quad k = 0,\pm1,\cdots\}.$$

4. <u>Exponential Abel-Euler Spline</u>. We consider the discrete exponential spline

$$\Phi_{n,h}(x,\lambda) = n!\left(\frac{\lambda}{\lambda-1}\right)^n \sum_{\nu=-\infty}^{\infty} \lambda^\nu \, Q^{\frac{h}{n}}_{n+1}(x-\nu)$$

which satisfies the functional equation

(4.1) $$\Phi_{n,h}(x+1,\lambda) = \lambda \, \Phi_{n,h}(x,\lambda).$$

The restriction of $\Phi_{n,h}(x,\lambda)$ to $[0,1]$ is $A_{n,\frac{h}{n}}(x,\lambda)$ and $\Phi_{n,h}(x,\lambda) \in S_n^{h/n}$.

By Theorem 2, if $h \leq \alpha < 1$, $\lambda \neq 1$, then $A_{n,\frac{h}{n}}(\alpha,\lambda) \neq 0$ if $\lambda \neq \lambda_\nu$, $\nu = 1,2,\cdots,n$, which are the zeros of $\Pi_{n,\frac{h}{n}}(\alpha,\lambda)$. This also proves (2.11) in view of (3.10) and (3.13).

Set

$$(4.2) \qquad S_{n,h}(x,\lambda) = \frac{\Phi_{n,h}(x,\lambda)}{\Phi_{n,h}(\alpha,\lambda)},$$

which belongs to the class $S_n^{h/n}$ and interpolates λ^x at $x = \alpha \pm \nu$, $\nu = 0,1,\cdots$.

We shall prove

THEOREM 3. *If* λ *is not a negative number,* $h \leq \alpha < 1$, *and* $S_{n,h}(x,\lambda)$ *is given by* (4.2), *then*

$$(4.3) \qquad \lim_{n \to \infty} S_{n,h}(x,\lambda) = |\lambda|^{x-\alpha} e^{iu(x-\alpha)}$$

where $\lambda = |\lambda| e^{iu}$.

For the proof of Theorem 3, we shall need a relation between the polynomials $A_{n,h}(x,\lambda)$ and the exponential Euler polynomials $A_n(x,\lambda)$ of Schoenberg [6] and the Fourier series representation of $\lambda^{-x} A_n(x,\lambda)$ which is 1-periodic.

From (3.3), it follows that

$$(4.4) \qquad \frac{\lambda - 1}{\lambda - e^z} e^{xz} = \sum_{n=o}^{\infty} \frac{A_n(x,\lambda)}{n!} z^n = \sum_{n=o}^{\infty} \frac{A_{n,h}(x,\lambda)}{n!} z^n e^{nzh}.$$

Hence it is easy to see that

$$(4.5)\qquad A_m(x,\lambda) = \sum_{n=o}^{m} \binom{m}{n} (nh)^{m-n} A_{n,h}(x,\lambda).$$

Inverting this relation (Riordan [4]) we have,

$$(4.6)\qquad A_{n,h}(x,\lambda) = \sum_{m=1}^{n} \binom{n}{m} (-1)^{n-m} h^{n-m}\, m\, n^{n-m-1} A_m(x,\lambda),$$

so that

$$(4.7)\qquad A_{n,\frac{h}{n}}(x,\lambda) = \sum_{m=1}^{n} \binom{n-1}{m-1} (-h)^{n-m} A_m(x,\lambda).$$

It is known (Schoenberg ([5], p. 399 (7.14)) that

$$(4.8)\qquad A_n(x,\lambda) = n!\, \frac{\lambda - 1}{\lambda}\, \lambda^x \sum_{-\infty}^{\infty} \frac{e^{2\pi kix}}{\lambda_k^{n+1}}\,, \quad 0 \le x \le 1,$$

where

$$\lambda_k = \log \tau + iu + 2\pi ik \quad (\lambda = \tau e^{iu}, \quad -\pi < u < \pi)$$

and $\lambda \neq 1$. Then for $0 \le x \le 1$, it follows from (4.2), (4.7) and (4.8), by easy manipulation that

$$\lambda^{\alpha-x} S_{n,h}(x,\lambda) = \frac{1 + \sum_{-\infty}^{\infty}{}' e^{2\pi kix} \left(\frac{\lambda_o}{\lambda_k}\right)^{n+1} \beta_{kn}}{1 + \sum_{-\infty}^{\infty} e^{2\pi ki\alpha} \left(\frac{\lambda_o}{\lambda_k}\right)^{n+1} \beta_{kn}}\,,$$

where

$$(4.9)\qquad \beta_{\kappa n} = \frac{P_{n-1}(h\lambda_k)}{P_{n-1}(h\lambda_o)}\,, \quad P_{n-1}(x) = \sum_{m=o}^{n-1} \frac{n-m}{m!}\, x^m\,,$$

and $\sum'$ denotes summation for all $k \neq 0$. It can be shown with some elementary but tedious manipulations which we omit that β_{kn} are uniformly bounded. Since $|\lambda_o/\lambda_k| < 1$, it follows that

$$\lim_{n\to\infty} S_{n,h}(x,\lambda) = |\lambda|^{x-\alpha}\, e^{iu(x-\alpha)}$$

where $\lambda = |\lambda|e^{iu}$. This completes the proof of Theorem 3.

5. Solution of Problem A. The solution of Problem A is now given by

THEOREM 4. *Let* $h \leq \alpha < 1$ *such that* $\Pi_{n,h}(\alpha,-1) \neq 0$. *Then for every set of data* $\{y_\nu\}$ *of power growth there is a unique function* $S(x) \in S_n^h$ *such that*

(5.1) $\quad S(\nu+\alpha) = y_\nu, \quad \nu = 0,\pm1,\pm2,\cdots,$

(5.2) $\quad S(x) = O(|x|^\gamma), \quad x \to \pm\infty$

where

$$|y_\nu| = O(|\nu|^\gamma) \quad \textit{as} \quad \nu \to \pm\infty$$

for some appropriate $\gamma \geq 0$.

The proof of the theorem follows the same lines as that of the corresponding result of Schoenberg ([6], p. 34) and will be omitted. It depends on the fundamental discrete spline $L(x) \in S_n^h$ such that

(5.3) $\quad L(\nu+\alpha) = \delta_{o\nu}, \quad \nu = 0,\pm1,\pm2,\cdots;$

(5.4) $\quad |L(x)| \leq Ae^{-B|x|}, \quad x \in \mathbb{R},$

for some positive constants A and B.

It is easy to see, following the ideas of Micchelli [3] that

$$L(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(x-\alpha)u}\,\Omega(x,u)du,$$

where

$$\Omega(x,u) = \frac{\displaystyle\sum_{k=-\infty}^{\infty} \frac{e^{2\pi ikx}}{\left(i(u+2k\pi)\right)^{n+1}}\, P_{n-1}\left(-ih(u+2k\pi)\right)}{\displaystyle\sum_{k=-\infty} \frac{e^{2\pi ik\alpha}}{\left(i(u+2k\pi)\right)^{n+1}}\, P_{n-1}\left(-ih(u+2k\pi)\right)}$$

and $P_{n-1}(x)$ is given by (4.9).

6. <u>t-Perfect Discrete Abel Splines</u>. Let m be a natural number and let $r = 0,\cdots,m$. For a given real number t we shall denote by

$$\mathcal{S}^r_{m,h,t} = \{S(x)\},$$

the class of all functions $S(x)$ with the following properties:

(6.1) The restriction $S_{m,\nu}(x)$ of $S(x)$ to $(\nu,\nu+1)$ represents a polynomial of degree m with the highest term $t^\nu x^m$,

(6.2) $S^{(k)}_{m,\nu}(\nu+1+kh) = S^{(k)}_{m,\nu+1}(\nu+1+kh)$, $k = 0,1,\cdots,r-1$,

where h is a positive number with $rh < 1$.

We now propose the following problem:

<u>Problem C</u>. *Determine* $S(x) \in S^r_{m,h,t}$ *having the least* t-*norm* $\|S\|_{t,\infty}$ *where*

$$\|S\|_{t,\infty} = \sup_{x \in \mathbb{R}} \left|\frac{S(x)}{t^{[x]}}\right| .$$

The analogous problem for $h = 0$ was proposed and solved in [7]. When $h \neq 0$, the easiest case which can be solved is when $r = m$. In this case we can easily prove following the argument of [7] *mutatis mutandis*

THEOREM 5. *Suppose* $m \geq 1$ *and* t *is a real number* $\neq 1$ *and* h *a given real number* $(mh < 1)$. *Then there is a unique discrete Abel spline* $F^*(x) \in S^m_{m,h,t}$ *which minimizes the* t-*norm where*

$$(6.3)\qquad \begin{cases} F^*(x) = A_{m,\frac{h}{m}}(x,t), & 0 \leq x \leq 1, \\ F^*(x+1) = tF^*(x), & \end{cases}$$

and $A_{m,h}(x,t)$ *is given by* (3.3) *with* λ *replaced by* t.

For $0 \leq r \leq m-1$, the problem is more difficult. We shall return to this problem later.

REFERENCES

[1] T. Lyche, Discrete polynomial spline approximation methods, Research Reports in Informatics, University of Oslo, Norway, 1975.

[2] O.L. Mangasarian and L.L. Schumaker, Best summation formulae and discrete splines, SIAM J. Numer. Anal., 10(1973), 448-459.

[3] C.A. Micchelli, Cardinal L-splines, Studies in Splines and Approximation Theory, Academic Press, 1976.

[4] J. Riordan, *Combinatorial Identities*, Wiley (New York): (1968).

[5] I.J. Schoenberg, Cardinal interpolation and spline functions IV, The exponential Euler splines, Linear Operators and Approximation ISNM, 20(1972), 382-404.

[6] _______________, Cardinal spline interpolation, SIAM Regional Conference Series 12(1973).

[7] A. Sharma and J. Tzimbalario, Cardinal t-perfect L-splines, SIAM J. Numer. Anal., 13(1976), 915-922.

[8] J. Tzimbalario, Cardinal Discrete Splines, to appear.

University of Alberta
Edmonton, Alberta, Canada
T6G 2G1

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 1746	2. GOVT ACCESSION NO. 14	3. RECIPIENT'S CATALOG NUMBER MRC-TSR-1746
4. TITLE (and Subtitle) DISCRETE EXPONENTIAL ABEL-EULER SPLINES. 6		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. N. Mohapatra and A. Sharma 10		8. CONTRACT OR GRANT NUMBER(s) 15 DAAG29-75-C-0024
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS #6 - Spline Functions and Approximation Theory
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE 11 April 1977 13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 19p.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

9 Technical summary rept.,

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cardinal splines

Abel interpolation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Cardinal discrete splines are modified by replacing the Lagrange interpolation of successive components by Abel interpolation.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

221 200